Exploring Links across Representations of Numbers with Young Children

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Numbers can be represented in a variety of ways – through pictures, diagrams, symbols. Each representation highlights different features of the number and the number system. This study aims to explore pupil understanding of number both within and across representations. A computer environment (suite of programmes) was created within which representations could be generated and manipulated. This study focuses on one of the programmes within which activities were developed for pupils in years 1, 2 and 3 of English primary schools (ages 5 to 8 years). The results were analysed across year groups and across attainment levels. The study found that not all representations are equally well understood. Reading figures accurately, often comes before an understanding of place value. Over the first three years of schooling there is improvement in understanding all representations although the number line and the beads seem to cause some difficulties. An ability to count in tens and ones is associated with greater understanding of many representations.

1 BACKGROUND

Over the last few years the main theme for discussion within mathematics education in England – particularly within the primary sector – has concerned pupils' performance with simple numerical operations. Much of the discussion emanated from the perception that the performance of English pupils was significantly worse than that of similar pupils in other countries (Harris, Keys and Fernades, 1997 Bierhoff, 1996). A term that is being used to encompass this discussion is Number Sense (Anghileri, 2000 McIntosh and Reys, 1992). McIntosh, Reys and Reys (1992) described this as:

> "…a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful strategies for handling numbers and operations" (p3).

In their framework for considering number sense they suggest that it is important for young children to develop multiple interpretations and representations of numbers. Further, researchers such as Gray, Pitta and Tall (1997), Tall (1993), Krutetski (1976), Carpenter, Fennema, Peterson, Chiang and Loef (1989), Carpenter, Fennema, Franke, Levi and Enpon (1999), Gravemeijer (1994), Thompson (1999) have all discussed the importance of representation in developing mathematical competence. Thompson (1999) and Menne (2001) point out that the way we represent mathematical concepts strongly affects the way in which we understand and develop such concepts and process numbers using our private mental methods. The National Numeracy Strategy (DfES, 1999) in England places a particular emphasis on attempting to shape and develop pupils' mental methods – particularly in the Early Years (5-7 years old). The importance of the link between number sense and representation of numerical ideas is summed up by Vergnaud (1987) who wrote:

> "Representation is a crucial element for a theory of mathematics teaching and learning, not only because the use of symbolic systems is so important in mathematics but also for two strong epistemological reasons 1) mathematics plays an essential part in conceptualising the real world; 2) mathematics makes a wide use of homomorphisms in which the reduction of structures to one another is essential." (p. 227)

Here, Vergnaud talks about symbol systems but this needs to be taken in its broadest sense since in mathematics we represent ideas/concepts through pictures, diagrams, tables, and graphs as well as by mathematical symbols. But the notion that the ways of representing mathematical ideas shapes and even constitutes our understandings of those ideas (Newton, 2000) is crucial and is something that needs to be addressed as young children start to explore mathematical ideas.

A study by Brenner, Herman, Ho and Zimmer (1999) explored the way 12 year old pupils from a number of countries (USA, Japan, Taiwan and China) made flexible use of representations in solving word problems. The problems which the pupils were asked to solve comprised two types of question: solution items (both simple straightforward calculations and word problems leading to a calculation), and a representation item corresponding to each solution item. There would be, for example, a solution item based on dividing fractions which the pupils could complete in one lesson and then in the following lesson the pupils would be given five to seven representations of the solution item and be asked to judge their validity. An example of the types of questions used is shown in Figure 1.

On the left hand side is the solution question and on the right is the corresponding representation question. In A2 the pupils need to comment on a variety of representations and consider whether they represent 3÷6 or not. In B3 all the representations are symbolic and again the pupils have to consider which are equivalent to the original sum. Brenner, Herman, Ho and Zimmer (1999) found that the Asian pupils outscored the American students on most items. But the more interesting results for our purposes is that the gap in performance was substantially greater in the representation items. This could indicate a greater conceptual awareness due to the ability to interpret the mathematical concepts both within and across a variety of representations. Clearly, more work needs to be undertaken to explore why this might be so for a particular culture but it seems that having an understanding of multiple representations of an idea/problem has a positive effect on the ability to solve the problem. Goldin and Shteingold (2001) suggest that representational systems are important to the learning of mathematics because of the inherent structure contained within each representation. This structure can shape or constrain learning. Further, different representations emphasise different aspects of a concept and so the development of an understanding of a particular concept comes from having a range of representations and being able to move between them.

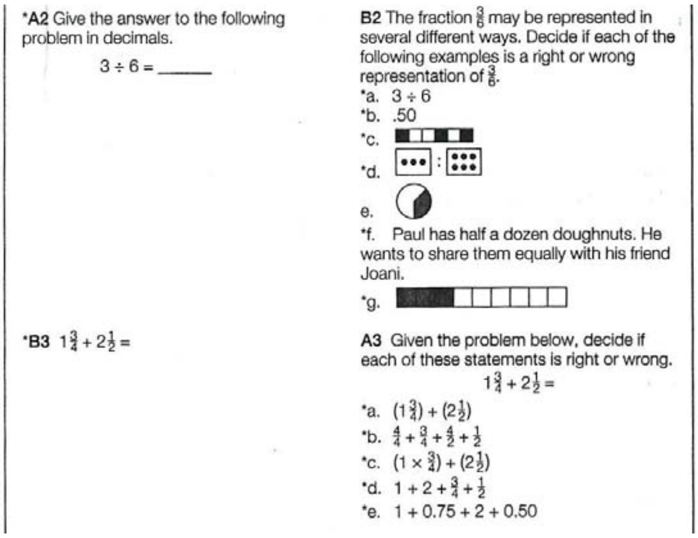

Figure 1 Exemplar questions

The place of representations within a theory of learning mathematics is discussed in some detail in Seeger (1998). He identifies a shift from the idea of a direct relationship between "the representation and what is being represented" (Seeger 1998 p. 312) to the Vygotskian idea of representations as mediators. If this view is pursued then representations cannot be seen as "beyond discussion" and "automatically grounding learning" (Seeger 1998 p. 334) but need to be considered as tools through which understanding can be constructed. The characteristics of the representation now become crucial and these characteristics become the basis for a discussion about meaning of particular concepts. Further the use of different representations can emphasise different aspects of a concept and moving between representations can deepen understanding.

When discussing representations it is sometimes considered appropriate to distinguish between internal and external representations - external representations being communally acceptable words, graphs, numerals etc. whereas internal representations are what happens at a very personal level. They may involve similar representations but they are personally derived as opposed to externally imposed. For the purposes of this study the emphasis is on external representations which may help pupils to develop flexible and powerful ways of working with concepts and operations. This resonates with the work of Seeger (1998) who suggests that representations need to be seen as "exploratory artefacts that allow the production of multiple perspectives on mathematical content" (p.337).

Essentially, representations are configurations of some kind that can represent something else in some manner (Goldin, 2002). There may be many ways of representing a concept and there will be a relationship between two or more configurations of the same concept. Representations actually represent each other. For example graphs represent equations and equations represent graphs. Goldin (2002) suggests that representations contain a number of characteristics.

First, they contain what might be called *primitive components* such as signs, symbols, objects. For young children this may include blocks, number cards or the digit themselves. Secondly, they contain *configurations* which are ways of combining the primitive components. These might be the way that blocks are arranged in order to represent numbers in such a way that place value is embedded in the configuration. Sometimes they contain *higher level structures* such as mathematical operations and rules which need to be adhered to in working with a particular representation. They also have *conventional and objective characteristics* so that once the conventions of a system have been agreed there are then specific characteristics that exist because of this. Thus once the conventions of a base 10 system are accepted then the properties of numbers within that are there to be found.

Kaput (1991) takes the view that the way the learner understands notation and representations determines the way in which mathematical thinking can develop. He suggests that mathematical notation acts in a similar way to the architecture of a building in that it constrains and/or supports our experience. Just as Tall (1993) discusses the importance of being able to see process and concept in the same expression, Kaput talks of the importance of being able to move from something being "form" at one level to it being "content" at another level. He maintains that the ability to see links between different representations (both iconic and symbolic) is a powerful problem-solving tool and he suggests that linking notational systems helps pupils to extend their reasoning processes from concrete to more abstract systems:

> "...all aspects of a complex idea cannot be adequately represented within a single notation system, and hence require multiple systems for their full expression……..multiple, linked representations will grow in importance as an application of the new dynamic interactive media" (Kaput, 1994, p. 530).

This is an ability clearly identified by the Russian thinkers (Krutetski, 1976) which is present in the thinking of the able mathematics pupil. Kaput suggests that linking notational systems helps pupils to extend their reasoning processes from concrete to more abstract systems. This is a theme that he explores further in his paper on Technology and Mathematics Education (Kaput, 1994) where he considers the way computer environments can be used to explore links between representations. In considering this he identifies three areas that need to be examined:

- *Dynamic versus static media*
- *Interactive versus inert media*
- *Procedure capturing and executing facility in an external device versus in human memory and cognition*

In static media "the states of the notational objects" (Kaput, 1994, p.525) are unchanged with time whereas in a dynamic medium this is no longer a restriction. Further, the change from one state to another is transparent and can be seen happening in a dynamic medium – less easy to see in a static medium. These intermediate stages could be important for scaffolding pupil development. A dynamic system allows more than one representation to be seen at the same time and changes in one representation can be seen to change the form of the other representation, thus enabling pupils to experience more than one representation of a particular concept. In considering the interactive nature of a computer environment the constraint and support elements of the environment need to be considered. For example, in working dynamically on the computer with Diennes blocks there is a clear constraint to work in powers of 10, but this is intended to act as support for the pupil as he/she builds up an understanding of place value within the denary system. (Diennes Blocks are a system of wooden cubes which are structured to represent place value so that in base 10, ten singles (1) make a long (10), ten longs make a flat (100) and ten flats make a large cube (1000)). Further, the interactive nature of the system requires user-directed inputs which can allow the system to operate in a variety of ways – operations can be stored and used later and intermediate stages can be shown or hidden depending on purpose, etc.

It is these aspects of a dynamic environment, particularly the first two, that we are interested in exploring within a suite of programmes, that is, a set of linked dynamic representations. The idea of multiple, linked representations as an aid to developing mathematical understanding, is the focus of this paper.

2 METHODOLOGICAL FEATURES OF THE WORK

2.1 Introduction

In order to explore pupil understanding of different representations a suite of computer programmes has been developed which allows the pupils to move between and operate within these different representations. The idea within these programmes is of children 'coming to know' not just 'coming to do'. Thus the programmes provide the opportunity for communication and discussion between pupils and between pupil and teacher about the work. Since the programme ideas make transparent the structural aspects of number work, the characteristics of different representations are highlighted and hence the role of representations as mediators for developing meaning can be developed.

In developing the suite of programmes there was a particular concern for pupils who find mathematics difficult and make slow progress. It was found in a previous study (Suggate, 1993) that children were able to carry out successfully more calculations with the visual representations than without. Further, after using the pilot programmes they were sometimes observed visualising the representations in their working. They would shut their eyes and use their fingers to point to imaginary diagrams. This resonates with the work of Gray, Pitta and Tall (1997) who found that for low attainers:

> "Imagery in the numerical context is strongly associated with the procedural aspects of numerical processes. The children carry out procedures in the mind as if they were carrying out procedures with perceptual items in front of them" (p.127).

They further observed that:

> "The ability to filter out information and see the strength of such a simple device as a mathematical symbol appears to be confined to the high achievers. The evidence suggests that children who are low achievers in mathematics appear unable to detach themselves from the search for substance and meaning – no information is rejected, no surface feature filtered out." (p.128).

The representations and imagery that the pupils are introduced to at an early stage could be crucial in facilitating their potential development. Hence the emphasis in the suite of programmes on mathematical structure.

2.2 Aims

The specific purpose of this project is to use a suite of IT programmes which will allow pupils, particularly those who are having difficulty with mathematics:

- to explore a variety of ways of representing numbers;
- to explore ways of undertaking addition and subtraction operations;
- to explore ways of undertaking multiplication and division operations.

In this paper mainly quantitative approaches have been used to:

- explore the ways that pupils relate to different representations and the way in which they are able to connect the representations;
- explore ways in which one of the programmes support young pupils as they work on their understanding of basic mathematical concepts.

2.3 The programme materials

The particular programme that was used with the pupils, Number 1-99, consisted of the set of representations of numbers as shown in Figure 2 below. These are not exhaustive but do represent the most common representations which young pupils meet. Noting the work of Goldin (2002), it can be seen that the representations have a variety of primitive components such as counters, blocks, matches and money and the configurations vary depending on the primitive components. For example, the unit blocks are structured so that ten blocks in a column form a single long block to emphasise the ten-ness in our number system and encourage pupils to move towards thinking both in tens and in units. Similarly, the beads are organised to emphasise the ten-ness of the system by using colour to distinguish groups of ten.

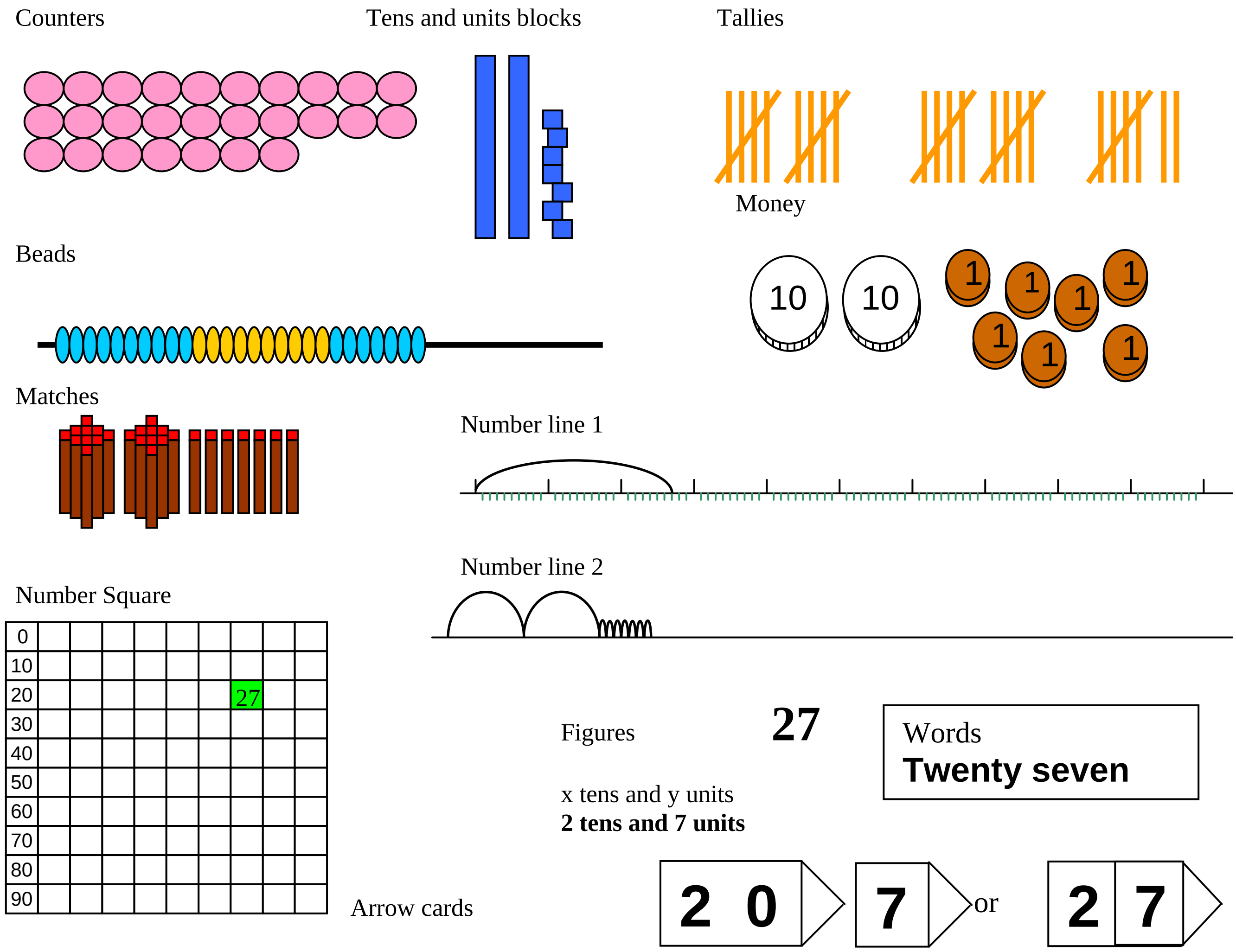

Figure 2 Number Representations used within the programme (up to 99) - Twenty seven is shown in each representation

2.4 The Sample

The data were collected from four schools in the North East of England – representing both urban and rural schools. In each school, 18 pupils were interviewed individually, 6 from each of the classes Year 1, Year 2 and Year 3 (6, 7 and 8 year olds respectively). In each class the teacher (based on her knowledge of the pupils) selected two top, two middle and two lower attaining pupils. Thus there was a sample size of 72 (24 in each of Years 1, 2, and 3, and 24 in each attainment group). The pupils were all very willing to help and appeared to enjoy using the computer. Originally data were collected from reception pupils (aged 5 years) also but the pupils found the exercise too difficult and that data were not used in this analysis.

2.5 Procedure

The pupils were introduced individually to the programme, *Number 1-99*. They were first shown the representations in Figure 2 on an A4 sheet and an explanation was given about how the number 27 could be shown in different ways. They were then asked to choose two representations. The researcher constructed a number in the first representation as shown in Figure 3 where the number 37 is constructed using tens and units blocks. The pupil constructed the same number in the other representation as shown in Figure 3 where 37 is constructed using the beads representation. In each case the pupil works in tens and units. This exercise was repeated twice with a different pair of representations each time, but with the pupil now constructing the number in both representations. Thus each pupil had direct experience of six representations and each time would see on the screen a pair of representations as shown in Figure 3.

Following this exposure to the representations and the programme itself, the pupils were shown a different number in each representation and asked to identify it. The order of the representations was randomly chosen for each pupil but the numbers used were the same each time. The pupils' answers were noted together with any relevant observation (such as, pointing, eye movement, vocalisation). The pupils were also asked how they had obtained their results. The accuracy of the answer was not the only focus of interest but any common errors were also noted as these might indicate where particular care is needed in teaching.

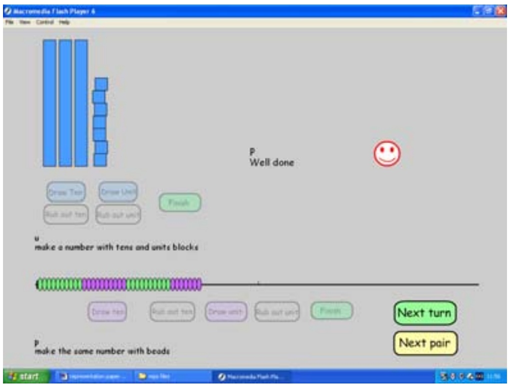

Figure 3 Example of the computer screen image

3 FINDINGS OF THE STUDY

The results are discussed in two parts. In the first section the analysis simply explores the results of the study using percentage scores. In the second section the analysis will be conducted in relation to a Rasch model. It can be seen from the graph (Figure 4) that in general the representations that generated the highest percentage of correct responses were counters, figures and arrow cards. Both the figures and the arrow cards look similar and, in a way, constitute a reading exercise and counters are probably the most common physical resource used by young pupils. The representations that gave rise to the most incorrect responses were beads and the number line 1. In neither case is there a numerical scaffold which would assist the pupils in determining the number. Interestingly, the representations which are least useful when it comes to representing operations are figures and arrow cards and one of the most commonly used representations for this purpose is the number line.

3.1 Percentage-based analysis

Figure 4 shows a graph of the overall results – showing correct responses, incorrect responses and no responses.

3.1.1 Results by Year Group

In order to explore the results in more depth three aspects of the data were explored. Firstly, the progression from Year 1 to Year 3 was considered, then the facility within attainment bands was considered and, finally, an analysis of the way in which the pupils worked was undertaken. The next graph (Figure 5) breaks down the results into year groups where the data has been used to create correct response profiles. The graph only shows the way in which the percentage of correct responses changes over the years.

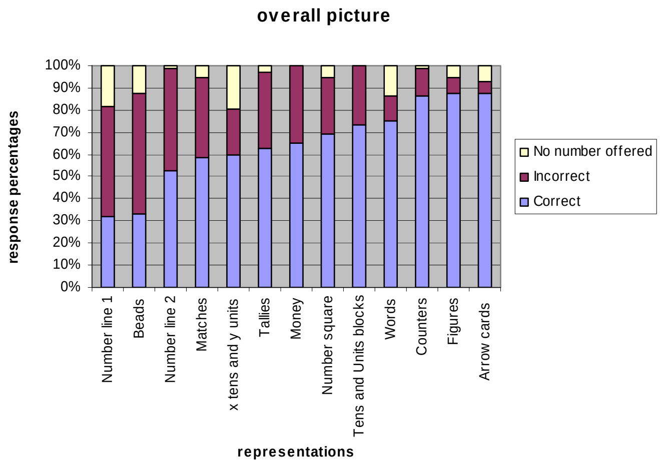

Figure 4 Overall results

As might be expected, the graph shows that pupils develop their competence rapidly in interpreting representations as they progress through the first few years of primary school. The two representations that stand out are again beads and the number line. Even in Year 3 these remain substantially less successful than the other representations. Further they are the only two representations for which there were still some non-responses in Year 3. It is interesting that the second form of the number line (see Figure 2) with two types of loops (for tens and units) was much more accessible. The loops provided very clear 10s and units cues for the pupils to work with.

In Year 1, the counters, the figures and the arrow cards (equivalent to figures) were the most successfully completed representations. Several of the children reached the correct answer for the counters by counting in ones (10 out of 16). 63% of the children were able to ‘read’ two digit numbers correctly, but the fact that no other representation was correctly interpreted by more than 33% of the children shows that at least half of those who could read the figures did not really understand the principles of place value. The difficulty of the traditional number line is clear. The results for Year 2 show very much the same pattern as Year 1. The easiest representations were figures (and arrow cards), counters, words, tens and units blocks and the number square (aided by the tens figures shown down the left hand side). The most difficult is again the number line with a single loop. This is possibly because there are really two representations here: the loop itself, and the marking on the line. In order to interpret the loop, the line marking has to be read.
The increased facility with the representations from Year 1 to Year 3 is shown in Table 1.

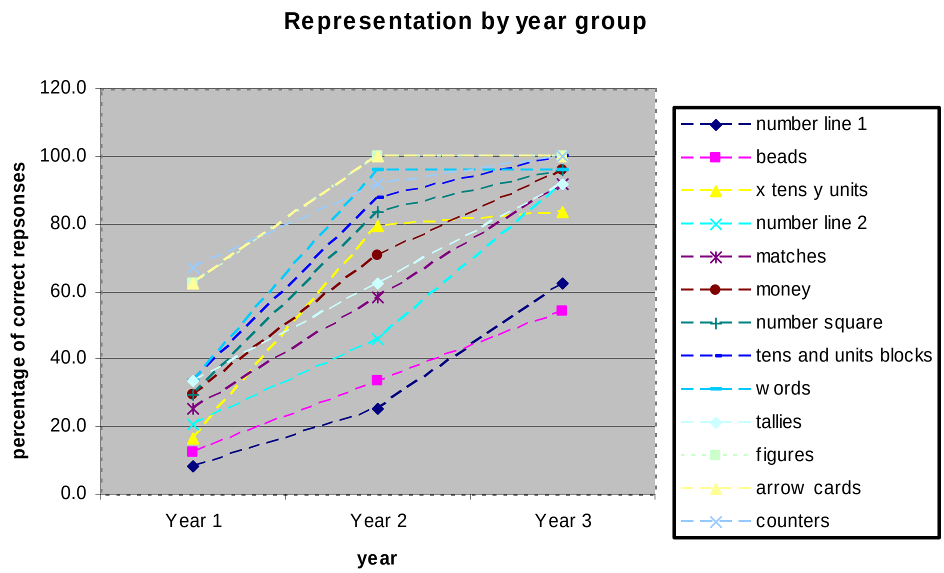

Figure 5 Results by year group

In the table it can be seen that, except for the number line which seems to be a conceptually more difficult representation for the pupils, facility with the representations increases substantially in the first year but is mostly the same or less in the second year. This is not unexpected since many of the representations could be considered to be 'school generated' in that pupils will meet the idea of representations such as this for the first time in school. These representations are a means of constructing understanding and are not simply mediators through which understanding occurs. Thus, pupils clearly have to work with these representations in order to understand how they represent ideas and operations.

representation	Increase (%) Year 1 to Year 2	Increase (%) year 2 to year 3
number line 1	*16.7*	*37.5*
beads	*20.8*	*20.9*
x tens y units	*62.5*	*4.1*
number line 2	*25.0*	*45.9*
matches	*33.3*	*33.4*
money	*41.6*	*25.0*
number square	*54.1*	*12.5*
tens and units blocks	*54.2*	*12.5*
words	*62.5*	*0.0*
tallies	*29.2*	*29.2*
figures	*37.5*	*0.0*
arrow cards	*37.5*	*0.0*
counters	*25.0*	*8.3*

Table 1 Increase in correct response facility from Year 1 to Year 3

3.1.2 Results by Attainment Groups

Secondly, the results were organised according to attainment groups in order to see if there was a pattern emerging from this about pupil approaches to representations. This data was used to produce the graph shown in Figure 6.

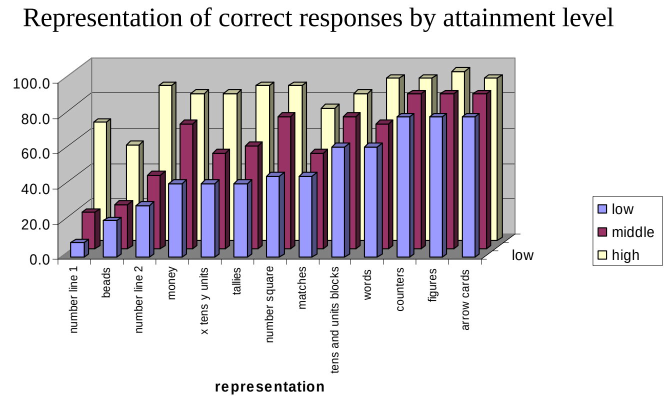

Figure 6 Results according to attainment level

This graph in Figure 6 shows quite clearly the difference in facility with representations between the low and high attaining pupils. With the high attaining pupils, the results show that generally they have a sound grasp of the range of representations although even here the two representations that have no numeric symbolic clues were the most difficult to interpret. With the low attaining pupils, counters, arrow cards and figures were clearly the representations which caused least difficulty: the arrow cards and the figures providing clear numeric-symbolic clues as to the number portrayed. Again the beads and both number lines caused the most difficulty.

Another feature which is illustrated in the graph in Figure 7 is the percentage of low attaining pupils who make no response. It can be seen here that for the high attaining pupils non-response was not an issue but it was particularly noticeable for the low attaining pupils. For four of the representations more than 20% of these pupils did not respond at all.

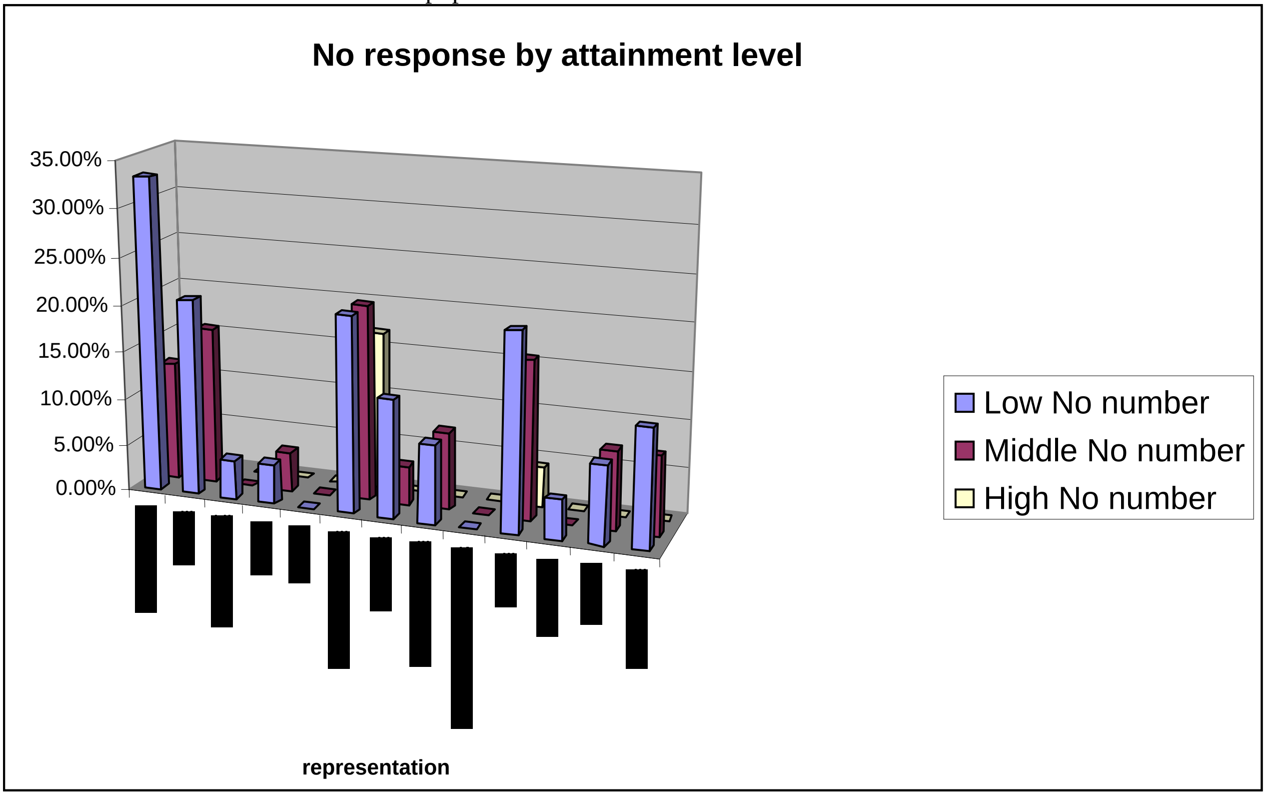

Figure 7 Incidence of non response by attainment group

3.1.3 Way of working

The way in which they performed the calculation was observed as the pupils worked on identifying the number for each representation. If necessary, the pupils were asked how they worked out the number. Essentially, there were two main methods: counting in ones and counting in tens. In order to analyse the way the pupils worked we divided the representations in Figure 2 into three groups. For each group a correct response factor was calculated by dividing the total number of correct responses by the total number of incorrect responses. The groups were as below.

Group 1. Grouping in tens clearly shown but counting in ones is still possible: counters / tens and units / tallies / matches / number line2 / beads

For example, with the beads, through the colours, representations of numbers are clearly grouped in tens and by focusing on colours the user can count in tens: however it is also possible to ignore the colours and just count the individual beads.

Group 2. Grouping is evident but counting in ones is not easy: money/x tens and y units/number line1

For example, with the money representation, pupils would know that ten 1p coins were the same value as a 10p coin but the representation does not really allow the learner to count very easily in ones. This is why, with low attaining pupils, they often will say that a representation that shows two 10p coins and three 1 p coins shows the number 5.

Group 3. Representations which are essentially conventions: figures/arrow cards/words/number square

Here there is no clear split into tens and units groupings. While the idea of place value is clearly evident, for example with arrow cards, it is not visually clear that ten of the unit cards "make" a tens card.

For the first group the graphs in Figures 8a,b and c show the correct response factor for the different ways that the pupils worked: it is clear here that for this group of representations the facility to work in groups of ten is very important. The correct response factor for working in 10s is 4.55 whereas the factor for working in 1s is 0.63. The factor for other methods is 0.7.

A similar exercise for the second group suggests that for this group the facility to work in tens is even more important with correct response factors of 9.83 for working with tens first, 0.1 for working with ones first and 0.92 for an alternative method. The main difficulty here was that for those pupils who were counting in ones, the 10p coins were just counted as one unit.

For the third group the results are quite different in that the pupils all used alternative methods. Usually they simply read the result from the representation.

For each of the groups, one of the key points was the link between accuracy of number identification and the facility to count in categories other than units, particularly in tens. In the first two groups, pupils who counted in ones were doing far more work than those who counted in tens and hence the opportunity for making errors was much greater.

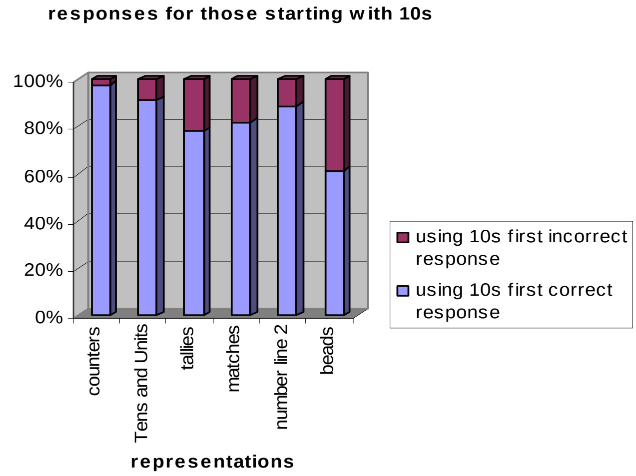

Figure 8a Correct/incorrect responses for group 1 (tens)

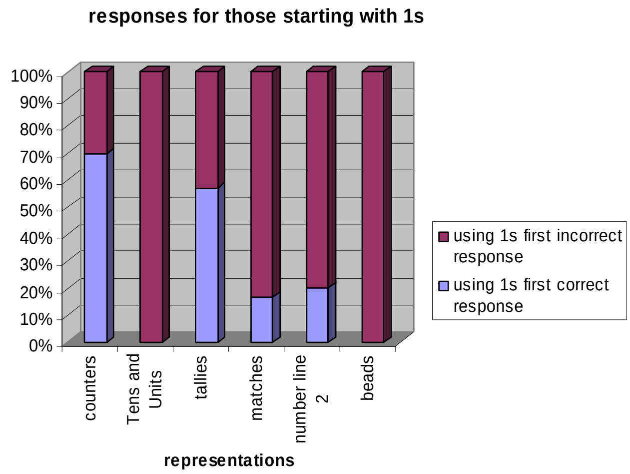

Figure 8b Correct/incorrect responses for group 1 (ones)

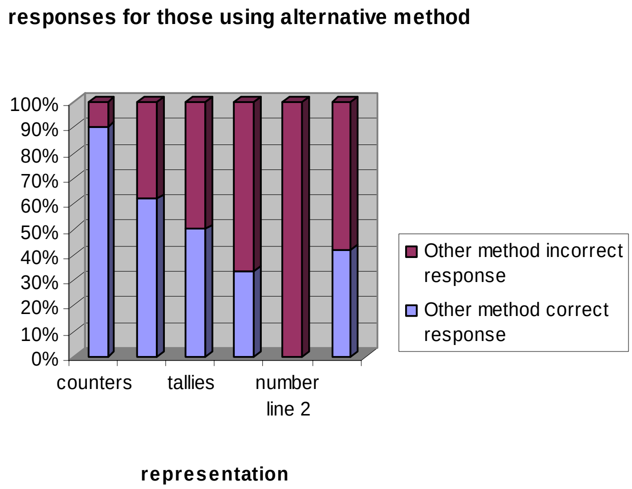

Figure 8c Correct/incorrect responses for group 1 (other)

3.1.4 Errors

It was possible to identify the probable cause of some of the mistakes. In almost all cases this showed an imperfect understanding of place value. It was noted that there seemed to be different types of errors with different representations. Any representation can only show some aspects of number. For example, the tens and units blocks show clearly how the ten units are the same as 1 ten block but, with the arrow cards, the tens and the units systems are, in a way, parallel systems with an understood link which is really implicit rather than explicit. So it is useful for children to experience and be able to use several different representations as they tend to focus on different aspects of the number. It was not always clear why pupils made errors or, indeed, how they made them. But, from the data collected, there were a number of types of errors that pupils made in trying to work out what number a particular representation showed.

The first error was where the pupils *counted all groups as 1 unit*. This occurred in four representations: tallies, matches, money and number line 2. Here for example the pupils who made this error would count the number of coins instead of the value of the coins shown, in the number line they would count the number of loops rather than the value of each loop. This error accounted for about 14% of all the errors (twenty pupils made the error once, five pupils made it twice, one pupil made it three times and one made it four times). This is what might be called a "value count" error where anything that appears to be an entity is counted as 1 unit. This is a conceptual error and clearly requires work on the nature of groupings and place value.

The second most common error was simply *a counting error* where pupils would miscount the number of beads or tallies and give a number which was 1 or 2 away from the correct answer. This accounted for about 13% of the errors and was mainly evident in the work of pupils who usually counted every representation in ones. This error could simply be the act of miscounting which is not necessarily a conceptual error.

Other errors that were made were representation specific. These included:

- *reversing the digits* which one pupil did for both figures and arrow cards;
- *adding the digits* which eight pupils did in the figures and tens and units representations;
- *giving an answer of 1* for the number on the number square since there was one number highlighted, which 4 pupils did.

3.2 Using a Rasch Model

The results above suggest that in this area of representation there are levels of difficulty that can be attached to the representations and that there are elements of progression in understanding the characteristics of the different ways in which numbers can be represented. One way to quantify this idea is to calculate Rasch measurements from the data. Essentially, a Rasch model is a measurement model which provides a basis for estimating a person's ability from that person's recorded performance on a set of items. The model proposes a mathematical relationship between a person's ability, the difficulty of the task, and the probability of the person succeeding on the task, (see for example Masters

(2001), and Bond and Fox (2001)). The equal interval scale of difficulty is the same for items and persons and the model permits a check on the unidimentionality of the scale. In this case, the items appeared to form a single unidimensional scale and this allows a plot on a single vertical scale for both pupil ability and item difficulty. Using the programme WINSTEPS (Linacre 2004), the Rasch scores were as on the chart in Figure 9.

This analysis also allows us to look at the differences between the year groups (Figure 10)

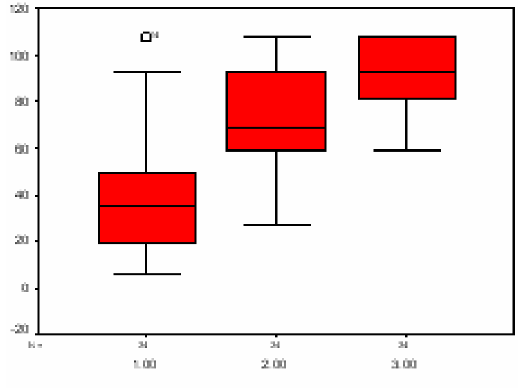

Figure 10 Box and whisker plots of Rasch scaled scores of pupils

These box and whisker plots show very clearly the development of the pupils as they progress from Year 1 to Year 3. It also shows the overlap across the years. The small box in the top left represents an outlier. Finally we can see the differences between ability groups in the means with error bars (Figure 11).

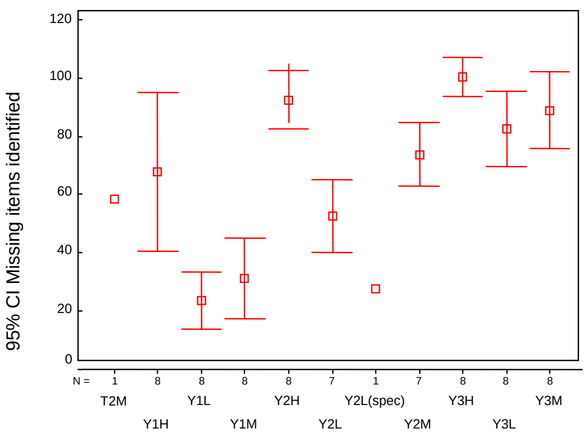

Figure 11 Measures with 95% Confidence intervals for Rasch scaled scores of classes

It can be seen that there is a substantial difference between the high attaining pupils and the other groups. In Year 1, these high attainers stand out as being the group who are able to make sense of the different representations with the other groups clearly having difficulties. In Year 2 progress is made by all groups but especially the middle attainers, whilst in the third year the greatest progress seems to be made by the low attainers. A conjecture from this could be that representation is a key concept that potentially gives pupils a sound foundation on which to build their mathematical development.

4 CONCLUDING COMMENTS

Essentially the two analyses suggest that there are five main findings:

- Not all representations are equally well understood (the number line seems to be especially difficult particularly for low attaining pupils). Indeed there is a clear hierarchy of difficulty associated with the representations used.
- ‘Reading figures’ accurately, which is a skill that virtually all pupils can accurately employ, comes before an understanding of place value.
- Over the first three years of schooling there is a great improvement in understanding all representations except the number line and the beads. There is a substantial difference between the responses of the high attainers and the other groups – particularly in the first year. Generally the greater change is through the first year.
- An ability to count in tens and ones is associated with greater understanding of many representations.
- Low attainers are not only liable to make more errors but they are also more likely to offer no response.

The number of children giving correct answers gives some indication of the difficulty of each representation and also possibly their prior experience. The representations in which the pupils showed most competence seem to be the counters, figures, arrow cards (effectively the same as figures), tens and units blocks and words. The more difficult ones were the number lines (especially the first, classic one in which the number was represented by a single arc), *x* tens and *y* units, and tallies. These results underline the point that children may be able to read figures correctly but still have little idea of the base ten structure of our number system. If the number line is to be used as a support for calculations (e.g. the empty number line model as used in the Netherlands (Gravemeijer, 1991), it needs to be recognised that for some pupils this is a representation that is not easy to understand. This empathises with Seeger (1998) who suggested that there was a danger of “representational overkill” (p.309) and that this had a “devastating influence” on the weaker students. He goes on to suggest that:

> “These students are supposed to benefit most from representations because it is assumed that what appears complex and complicated to them is made easier by projecting it onto a plane of the seemingly simple and obvious language of pictures, graphs, schematic diagrams, and the like” (p.309).

In a sense this is the wrong argument as we are not making ideas/concepts easier, but what we are trying to do is to help pupils gain deeper understanding of the ideas. It may well be that these representations have to be viewed as abstract conceptual frameworks through which pupils can construct understandings of number and operations and not just as illustrative mediators. This is in agreement with the later ideas of Seeger that the representations need to be seen as tools through which understanding can be constructed.

```
           <more>|<rare>
 100         ##### +
                   |
                   |
                   |
                   |T
                   |
  90       .####### +
                   |
                  S| Nline1
                   | Beads
                   |
                   |
  80               +
             #### |
                   |
                   |
                   |
                   |S
  70            ## +
                   |
                   | Nline2
                   |
              ### |
                   |
  60               M+ Matches
              .## | Xtens
                   |
                   | Tallies
                   |
                 # |
  50               +M Money
                   |
                 # | Nsquare
                   |
                   |
                .# | T&U
  40               +
                   | Words
                   |
                 # |
                   |
                  S|
  30               +
                ## |S
                   |
                   |
                   |
              .## | Counters
  20               + Acards   Figures
                   |
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                   |
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                   |
  10           ### +
           <less>|<frequ>
 EACH '#' IS 2.
```

Pupil success

Level of difficulty

This shows on the right-hand side the levels of difficulty of the different forms of the test (items) and on the left-hand side the success for the children. This has been fixed to have a mean score of 50 as the average score of the item difficulties and the pupils are arraigned on the left-hand side according to their scores. The children were in general more able than the items. The average for the children was about 60 and about 20 of the children were more able than the hardest item.

The scale shows practically the same order that is shown in Figure 4 above. The advantage here is that it shows the data on an equal interval scale and errors of measurement are available for each item and for each child. It shows that Nline1 and Beads are the hardest and are of identical difficulty and that Counters, Acards and Figures are also of identical difficulty and are the easiest. The rest are organised on the scale between those extremes.

The errors for each of the items is about the same which means that we can have equal confidence in the position of each item in the chart above.

The Winsteps programme gives tables to check whether items in the test fit the model or not. Broadly speaking the items fit the data very well indeed. The infit statistics, which check responses to items close to one another, show the closest fit. The outfit statistics suggest quite a close fit but a little less perfect.

A principal components analysis of the items shows that more than 95% of the variance was explained by the measures and just 0.6% by an additional factor.

NB on the scale shown, M represents the mean, S represents one standard deviation from the mean and T represents two standard deviations from the mean.

Figure 9 Map of persons and items difficulties

It is also in line with the work of Mason (2002)

> " … to get much educational benefit, students need to be active in processing images; they need to work on images, not just look at them. They need to probe beneath surface reactions. Working on and with mental imagery supports this development" (p.78).

The results above suggest that a key element in the development of early mathematics programmes could be the facility to work with different representations of numbers. In this case, it could be that it is through the ability to see numbers through different representations that pupils are able to build up their understanding of number concepts. The facility to see through the representations can help pupils to build their sense of a number and numbers in general. Figure 6 indicates that high attainment and the facility to be able to interpret representations are connected. This is compatible with the studies discussed earlier by Brenner, Herman, Ho and Zimmer (1999) and Zaskisand and Gadowsky (2001), and a study by Harries, Barrington and Hamilton (2002). How the connection happens needs more careful consideration since it could be argued that high attaining pupils have a better understanding of number and are therefore more able to make sense of representations than other pupils whose understanding of number is more limited. But an early assessment of pupils' facility with different representations could identify key areas for development. The computer has facilitated the identification of particular aspects of mathematical development and the exploration of pupil understanding.

The work points to two dimensions that need further attention:

1. A conceptual dimension which consists of the big ideas of counting and conservation, visualisation, addition/subtraction, and multiplication/division
2. A number domain dimension which consists of a set of number domains within which the concepts are explored.

The implications from the study are:

- Representations (particularly number lines) are not immediately transparent to the learner. This was paralleld in the failure of attempts to have older students use such representations in a study by Wooler (2004). As a support for calculation in particular the representations will need substantial teacher input and learner participation in order to understand the benefits and uses of the representation. This takes time and effort in order that the learner can both learn to notice the essential features of a representation and then use those features in order to manipulate numbers accurately
- Reading figures correctly does not imply understanding of the place value system. Children can read or recognise a number (for example, bus numbers and house numbers) without an appreciation of its structure in much the same way as children can read words without appreciating the phonic structure. Again working on the representations can help to illuminate the essential structure of the numbers and the number system.
- It is not clear which comes first, the facility to count in tens and ones, or the facility to use many different number representations. But it does appear that an awareness of the nature of groupings is the central feature of place value. This is a key idea within, for example, the Hungarian mathematics curriculum (Sutherland, Harries and Winter, 2001) where a whole range of activities are devised in order to help children appreciate the power of particular groupings.

Young children clearly vary greatly in their mental development and learning. It is likely, therefore, that introducing them to a manageable range of representations would help them experience their power and, maybe, their limitations. Further the process of moving mentally from one representation to another could involve "abstracting" the common principle of grouping. How this is to be done is not clear but, at the very least a permeating theme of the mathematics curriculum should be: what representations can be developed to illustrate a particular concept? What features of the concept do particular representations highlight? How do these representations allow us to manipulate ideas within a concept?

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